A New Finite-Difference Time-Domain Algorithm For Solving Maxwell's Equations

Zhiqiang Bi, Keli Wu, Chen Wu, and John Litva

Abstract—**A new algorithm is presented for deriving finite-difference time-domain (FD-TD) solutions of Maxwell's equations. When compared with Yee's method, it is found that the stability conditions for this method exceed those of Yee's method by the factors 1.41 and 1.73, respectively, for the two-dimensional and three-dimensional cases. Two additional important advantages of the method are given in the conclusions.**

I. Introduction

To date, Yee's FD-TD method has received a great deal of attention because it has a number of desirable attributes, such as the ability to analyze the complex microstrip antenna structures [2]. Recently, several other time domain methods have been presented. They have been developed to overcome the rectangular lattice limitation of the method. These include finite element derived methods [3] and a point-matched time domain finite element method [4]. All of these new methods make use of the conforming ability of the finite-element method to approximate physical boundaries more accurately.

This letter proposes a modified form of the finite-difference algorithm, with a stability condition which exceeds that of Yee's FD-TD method. As well, this method has several other advantages. The main issues underlying this technique are presented in the following sections.

II. Description of the New Algorithm

For ease of understanding, we introduce the algorithm by describing the two-dimensional case, and assume TM wave propagation. Under these conditions, Maxwell's equations become

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial y} \tag{1}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\frac{\partial E_z}{\partial x} \tag{2}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right). \tag{3}$$

The equations can be expressed in the finite-difference form by using a new algorithm, which is analogous to the rotated Richtmyer algorithm [5], [6]. For example, the first takes the form:

$$H_x^{n+\frac{1}{2}}(i,j) = H_x^{n-\frac{1}{2}}(i,j) - \frac{\Delta t}{\mu \Delta y} \cdot \left\{\frac{1}{2}\left[E_z^n(i,j) + E_z^n(i,j-1)\right] - \frac{1}{2}\left[E_z^n(i-1,j) + E_z^n(i-1,j-1)\right]\right\}. \tag{4}$$

The lattice used for implementing the proposed algorithm is shown in Region A of Fig. 1. It differs from the conventional lattice used for Yee's method, which is shown in Region B of Fig. 1. From (4), we can see that the scheme consists of two steps. The first step consists of finding average values for the components of the fields on fictitious nodes such as those at p, p' and q, q'. During the second step one uses the values obtained in the first step to derive the centered difference approximation to Maxwell's equations. This method is very compatible with Yee's method. The results from a numerical experiment are given in the following section to demonstrate the advantages of this technique.

III. Accuracy and Stability of the Method

It can be proven [7] that, when we use the new scheme to approximate a differential equation, the principal part of the local truncation error due to the finite differences approaches zero with the second order of the mesh lengths Δt and h. Hence, the new algorithm is of second order accuracy. Using a method similar to that used by Wilson [5], it is proven [7] that the new scheme is stable if

$$\frac{c(\Delta t)}{\Delta x} \leq 1, \tag{5}$$

where c is the velocity of propagation, and $\Delta x = \Delta y$. This stability criterion is independent of the number of dimensions if the computational grid is uniform. This invariance with the dimensions of the problems is considered to be an advantage of our algorithm because it is not enjoyed by Yee's technique. Yee's stability condition depends on the number of dimensions as

$$\frac{c\Delta t}{\Delta x} \leq \frac{1}{\sqrt{n}}. \tag{6}$$

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The authors are with the Communications Research Laboratory, McMaster University, Hamilton, ON, Canada L8S 4K1.

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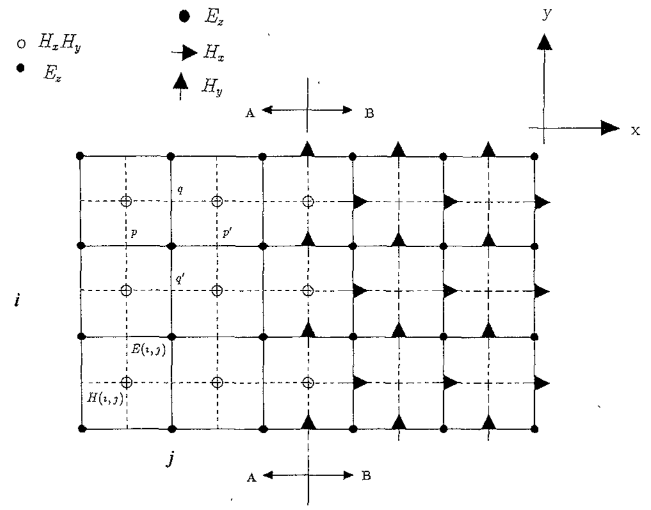

Fig. 1. FD-TD two-dimensional lattice showing TM field placement.

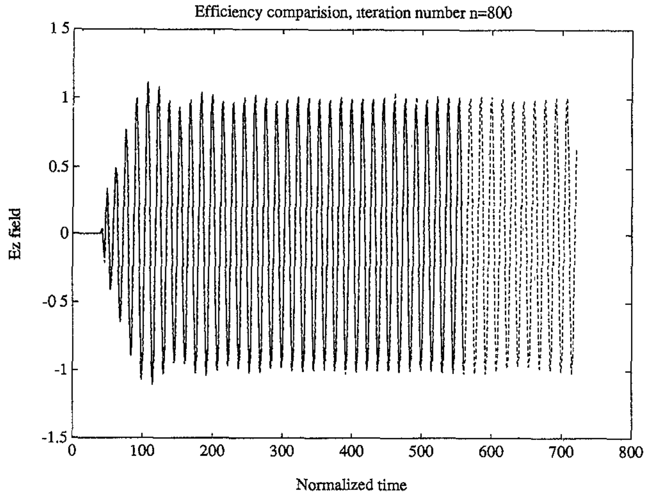

Fig. 2. Efficiency comparison of Yee's method and the new method.

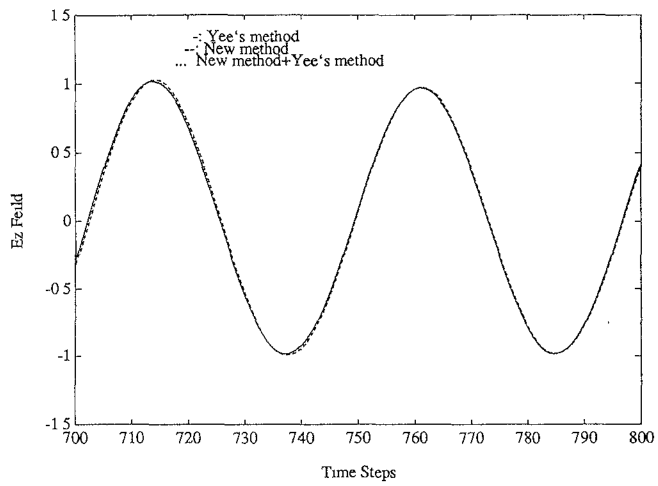

Fig. 3. Compatibility of the new method and the Yee's method.

We have carried out a numerical experiment to test the stability condition given by (5), as well as to check the validity of the new algorithm. An H-plane rectangular waveguide is chosen as the example [8] for our computation, where the excitation that is used on the excitation plane consists of a monochromatic dominant TE_{10} mode wave of unit amplitude. Both Yee's method and the new method have been applied to this problem. The stability factor, defined by $\rho = \frac{c(\Delta t)}{\Delta x}$, is assumed to 0.70 and 0.990, respectively, for the former and latter algorithms. A comparison of the results obtained using these two techniques is given in Fig. 2. The quantity that is being compared is the E_z field at a reference point. It is readily seen that at iteration 800 the new technique provides results covering a greater amount of time than the old technique. It is in this sense that the new method is considered to be more efficient than Yee's method. Another computation was carried out to test the stability conditions for Yee's method and the new method, respectively. The results show that once the stability factor exceeds 0.7071068, Yee's method begins to diverge. But, in the case of the new method, the algorithm does not start to diverge until the value of stability factor exceeds 1. In Fig. 3 is given the compatibility testing result of the new method and Yee's method. In this test, the waveguide, as was the case in Fig. 1, consist of two regions, A and B. In Region A, we use the new scheme, and in Region B, we use the Yee's method. The field is sampled in Region B. From this figure, we can see that the two methods are very compatible.

IV. The Relationship Between the Finite-Element Derived Time-Domain Methods (FE-TD) and the New FD-TD Method

It is shown [7] that the finite-element derived time-domain method [4], defined over a rectangular subspace and formulated using isoparametric functions, is equivalent to the new finite-difference time-domain method.

V. Conclusion

A new finite-difference time-domain algorithm for solving Maxwell's differential equations is presented. The accuracy of this new algorithm is the same as Yee's method, i.e., second order accuracy in both the time and space domains. The most important advanatge of the method, compared with the Yee's conventional method, is in the value of its stability condition. The stability condition for the new FD-TD exceeds that for Yee's method by factors 1.4 and 1.73 for the two-dimensional and three-dimensional cases, respectively. As well, there are two other important advantages of this method.

First, the method is compatible with both Yee's FD-TD method and the recently developed finite-element time-domain method. With the help of the new method, the conventional FD-TD and the newer FE-TD methods can be unified. One immediate benefit that can be realized from unifying the FD-TD and FE-TD is that the conforming boundary element method presented by Cangellaris [4] can be simplified in the following manner: near the structure, quadrilateral elements are used to conform the physical boundary, but away from the structure, the conventional grid of Yee's method is used.

Second, the new FD-TD method will provide greater flexibility for formulating and studying the multigrid method, variable mesh method and the method of finite difference approximations of the boundary conditions. Further work on these improvements to the FD-TD technique will be carried out in the near future.

Before concluding, it should be mentioned that although the stability condition for the new FD-TD technique is considerably improved over that of Yee's method, the total computation efficiencies of the two methods are almost the same on most current computers. With the development of parallel computation, the new method has strong potential for increasing the efficiency of FD-TD methods because it requires fewer time steps in solving a particular problem, i.e., less communication is required.

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